Principles of Robot Autonomy I

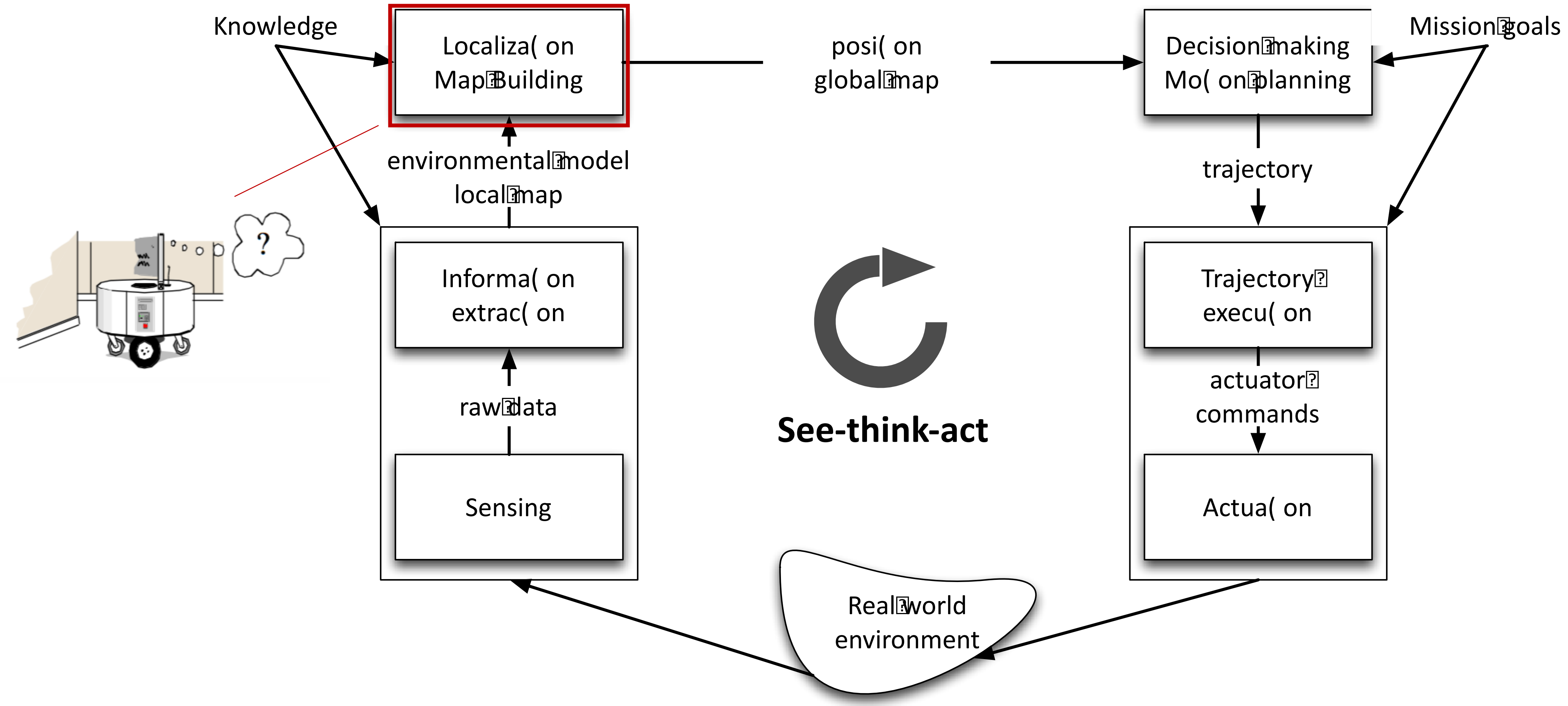
Introduction to state estimation and filtering theory

Stanford University

Autonomous Systems Lab

Attendance Form

The see-think-act cycle



Agenda

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  - Basic concepts about Bayesian filtering

- Readings:
  - Chapter 13, sections 13.1 – 13.3 in D. Gammelli, J. Lorenzetti, K. Luo, G. Zardini, M. Pavone. *Principles of Robot Autonomy*. 2026.

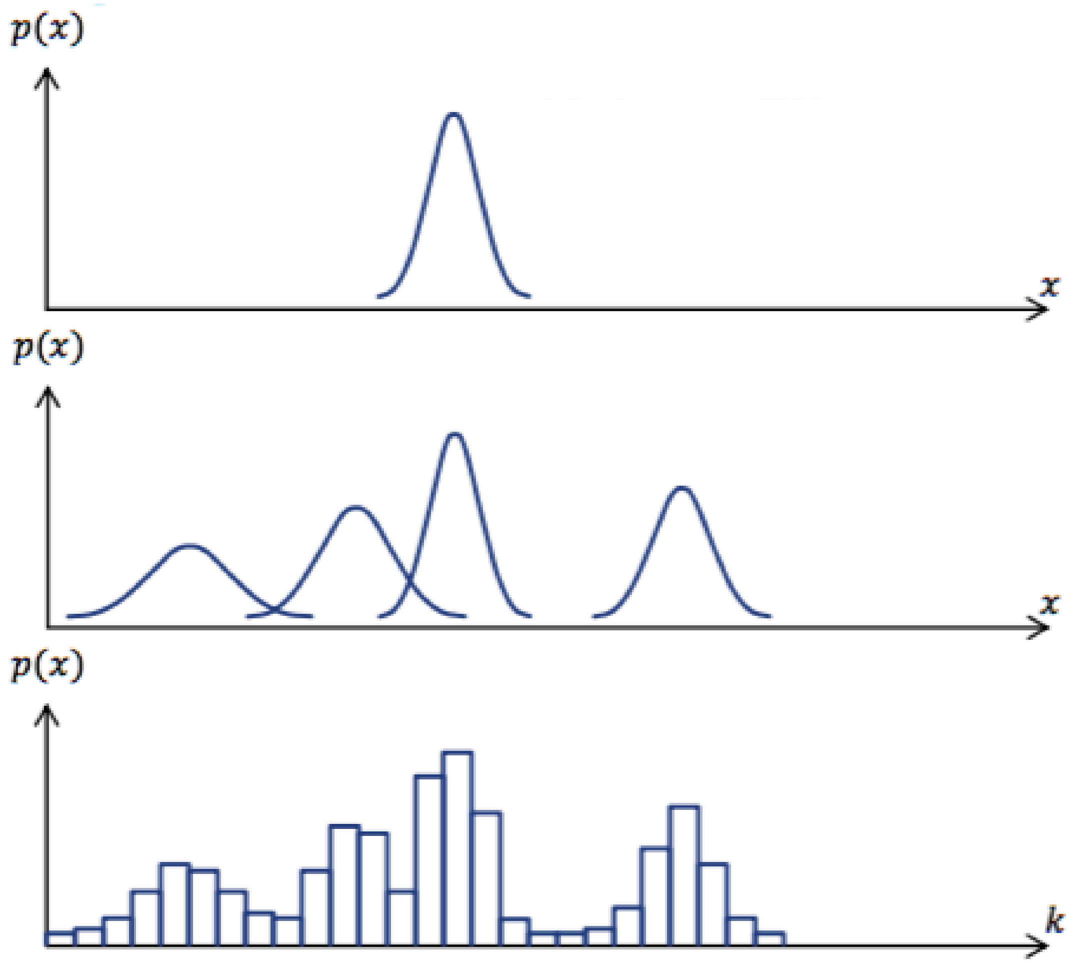
Map-based localization

- Key idea: robot *explicitly* attempts to localize by collecting sensor data, then updating belief about its position with respect to a map

- Two main aspects:
  - *Map representation*: how to represent the environment?
  - *Belief representation*: how to model the belief regarding the position within the map?

Probabilistic map-based localization

- Key idea: represent belief as a probability distribution
    1. Encodes sense of position
    2. Maintains notion of robot’s uncertainty
- Belief representation:
    1. Single-hypothesis vs. multiple hypothesis
    2. Continuous vs. discretized
- Today we will overview basic concepts in Bayesian filtering



Basic concepts in probability

- Key idea: quantities such as sensor measurements, states of a robot, and its environment are modeled as random variables (RVs)
- Discrete RV: the space of all the values that a random variable *X* can take on is *discrete*; characterized by probability mass function (pmf)

$$\underbrace{p(X}_{\text{Random variable}} = \underbrace{x)}_{\text{Specific value}} \quad (\text{or } p(x)), \qquad \sum_x p(X = x) = 1$$

- Continuous RV: the space of all the values that a random variable *X* can take on is *continuous;* characterized by probability density function (pdf)

$$P(a \leq X \leq b) = \int_a^b p(x)\, dx, \qquad \int_{-\infty}^{\infty} p(x)\, dx = 1$$

Joint distribution, independence, and conditioning

- Joint distribution of two random variables X and Y is denoted as

$$p(x, y) := p(X = x \text{ and } Y = y)$$

- If X and Y are independent

$$p(x, y) = p(x)p(y)$$

- Suppose we know that $Y = y$ (with $p(y) > 0$); conditioned on this fact, the probability that the X's value is x is given by

$$p(x \mid y) := \frac{p(x, y)}{p(y)}$$

Conditional probability

Note: if X and Y are independent

$$p(x \mid y) := p(x)!$$

Law of total probability

- For discrete RVs:

$$p(x) = \sum_y p(x, y) = \sum_y p(x \mid y) p(y)$$

- For continuous RVs:

$$p(x) = \int p(x, y) dy = \int p(x \mid y) p(y) dy$$

- Note: if $p(y) = 0$, define the product $p(x \mid y) p(y) = 0$

Bayes' rule

- Key relation between $p(x \mid y)$ and its "inverse," $p(y \mid x)$
- For discrete RVs:

$$p(x \mid y) = \frac{p(y \mid x)p(x)}{p(y)} = \frac{p(y \mid x)p(x)}{\sum_{x'} p(y \mid x')p(x')}$$

- For continuous RVs:

$$p(x \mid y) = \frac{p(y \mid x)p(x)}{p(y)} = \frac{p(y \mid x)p(x)}{\int p(y \mid x')p(x')\, dx'}$$

Bayes' rule and probabilistic inference

- Assume *x* is a quantity we would like to infer from *y*
- Bayes rule allows us to do so through the inverse probability, which specifies the probability of data *y* assuming that *x* was the cause

Posterior probability distribution

Prior probability distribution

$$p(x \mid y) = \frac{p(y \mid x)p(x)}{\int p(y \mid x')p(x')\,dx'}$$

Data

Normalizer, does not depend on $x \coloneqq \eta^{-1}$

- Notational simplification

$$p(x \mid y) = \eta\, p(y \mid x)p(x)$$

More on Bayes' rule and independence

- Extension of Bayes rule: conditioning Bayes rule on *Z=z* gives

$$p(x \mid y, z) = \frac{p(y \mid x, z)p(x \mid z)}{p(y \mid z)}$$

- Extension of independence: *conditional independence*

$$p(x, y \mid z) = p(x \mid z)p(y \mid z), \qquad \text{equivalent to } \begin{cases} p(x \mid z) = p(x \mid z, y) \\ p(y \mid z) = p(y \mid z, x) \end{cases}$$

- Note: in general

$$p(x, y \mid z) = p(x \mid z)p(y \mid z) \not\Rightarrow p(x, y) = p(x)p(y)$$

$$p(x, y) = p(x)p(y) \not\Rightarrow p(x, y \mid z) = p(x \mid z)p(y \mid z)$$

Expectation of a RV

- Expectation for discrete RVs: $E[X] = \sum_x x\, p(x)$
- Expectation for continuous RVs: $E[X] = \int x\, p(x)\, dx$
- Expectation is a linear operator: $E[aX + b] = a\, E[X] + b$
- Expectation of a vector of RVs is simply the vector of expectations
- Covariance

$$\mathrm{cov}(X, Y) = E[(X - E[X])(Y - E[Y])^T] = E[XY^T] - E[X]E[Y]^T$$

Model for robot-environment interaction

- Two fundamental types of robot-environment interactions: the robot can influence the state of its environment through control actions, and gather information about the state through measurements
- State x_t: collection at time t of all aspects of the robot and its environment that can impact the future
  - Robot pose (e.g., robot location and orientation)
  - Robot velocity
  - Locations and features of surrounding objects in the environment, etc.
- Useful notation: $x_{t_1:t_2} := x_{t_1}, x_{t_1+1}, x_{t_1+2}, \ldots, x_{t_2}$
- A state x_t is called *complete* if no variables prior to x_t can influence the evolution of future states → Markov property

Measurement and control data

- Measurement data z_t: information about state of the environment at time t; useful notation

$$z_{t_1:t_2} := z_{t_1}, z_{t_1+1}, z_{t_1+2}, \ldots, z_{t_2}$$

- Control data u_t: information about the change of state at time t; useful notation

$$u_{t_1:t_2} := u_{t_1}, u_{t_1+1}, u_{t_1+2}, \ldots, u_{t_2}$$

State equation

- General probabilistic generative model

$$p(x_t \mid x_{0:t-1},\, z_{1:t-1},\, u_{1:t})$$

Convention: first take control action and then take measurement

- Key assumption: state is complete (i.e., the Markov property holds)

$$p(x_t \mid x_{0:t-1},\, z_{1:t-1},\, u_{1:t}) = p(x_t \mid x_{t-1}, u_t)$$

State transition probability

- In other words, we assume *conditional independence*, with respect to conditioning on x_{t-1}

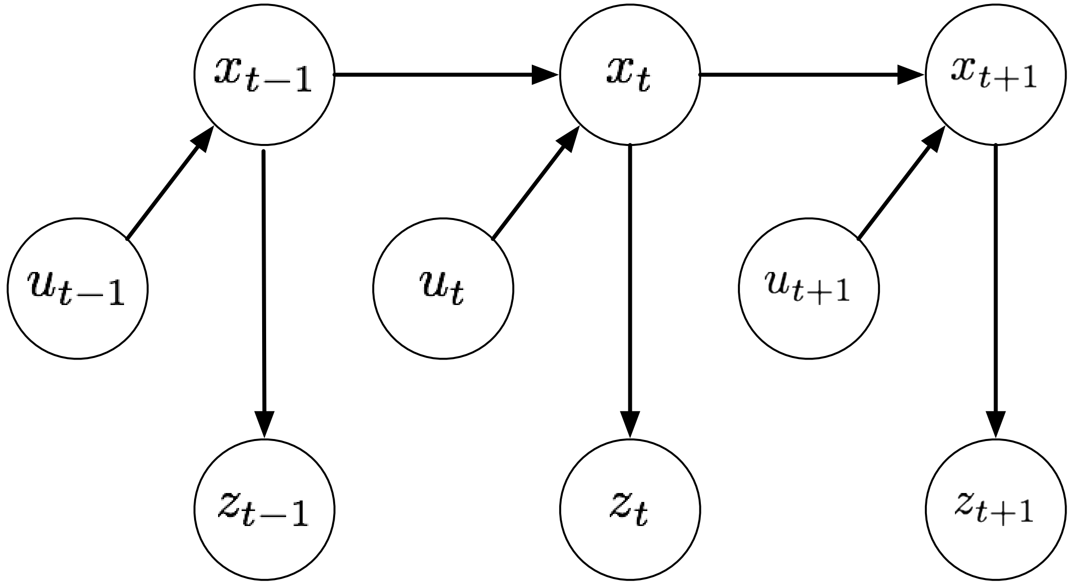
Measurement equation and overall stochastic model

- Assuming x_t is complete

$$p(z_t \mid x_{0:t}, z_{1:t-1}, u_{1:t}) = p(z_t \mid x_t)$$

Measurement probability

- Overall dynamic Bayes network model (also referred to as hidden Markov model)



Belief distribution

- Belief distribution: reflects internal knowledge about the state
- A belief distribution assigns a probability to each possible hypothesis with regard to the true state
- Formally, belief distributions are posterior probabilities over state variables conditioned on the available data

$$bel(x_t) := p(x_t \mid z_{1:t},\, u_{1:t})$$

- Similarly, the *prediction* distribution is defined as

$$\overline{bel}(x_t) := p(x_t \mid z_{1:t-1},\, u_{1:t})$$

- Calculating $bel(x_t)$ from $\overline{bel}\,(x_t)$ is called correction or measurement update

Bayes filter algorithm

- Bayes' filter algorithm: most general algorithm for calculating beliefs
- Key assumption: state is complete

- Recursive algorithm
  - Step 1 (prediction): compute $\overline{bel}\,(x_t)$
  - Step 2 (measurement update): compute $bel(x_t)$
- Algorithm initialized with $bel(x_0)$ (e.g., uniform or points mass)

Update rule

Data: $bel(x_{t-1}), u_t, z_t$
Result: $bel(x_t)$
foreach x_t **do**

$$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})\, dx_{t-1};$$

$$bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t);$$

end
Return $bel(x_t)$

Derivation: measurement update

$$
\begin{aligned}
bel(x_t) &= p(x_t \mid z_{1:t},\, u_{1:t}) \\
&= \frac{p(z_t \mid x_t, z_{1:t-1},\, u_{1:t})\, p(x_t \mid z_{1:t-1},\, u_{1:t})}{\underbrace{p(z_t \mid z_{1:t-1},\, u_{1:t})}_{:=\eta^{-1}}} && \text{Bayes rule} \\
&= \eta\, p(z_t \mid x_t)\, \underbrace{p(x_t \mid z_{1:t-1},\, u_{1:t})}_{=\overline{bel}(x_t)} && \text{Markov property}
\end{aligned}
$$

Derivation: correction update

$$\overline{bel}(x_t) = p(x_t \mid z_{1:t-1}, u_{1:t})$$

$$= \int p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t})\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1}$$

Total probability

$$= \int p(x_t \mid x_{t-1}, u_t)\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1}$$

Markov

$$= \int p(x_t \mid x_{t-1}, u_t)\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})\, dx_{t-1}$$

For general output feedback policies, u_t does not provide additional information on x_{t-1}

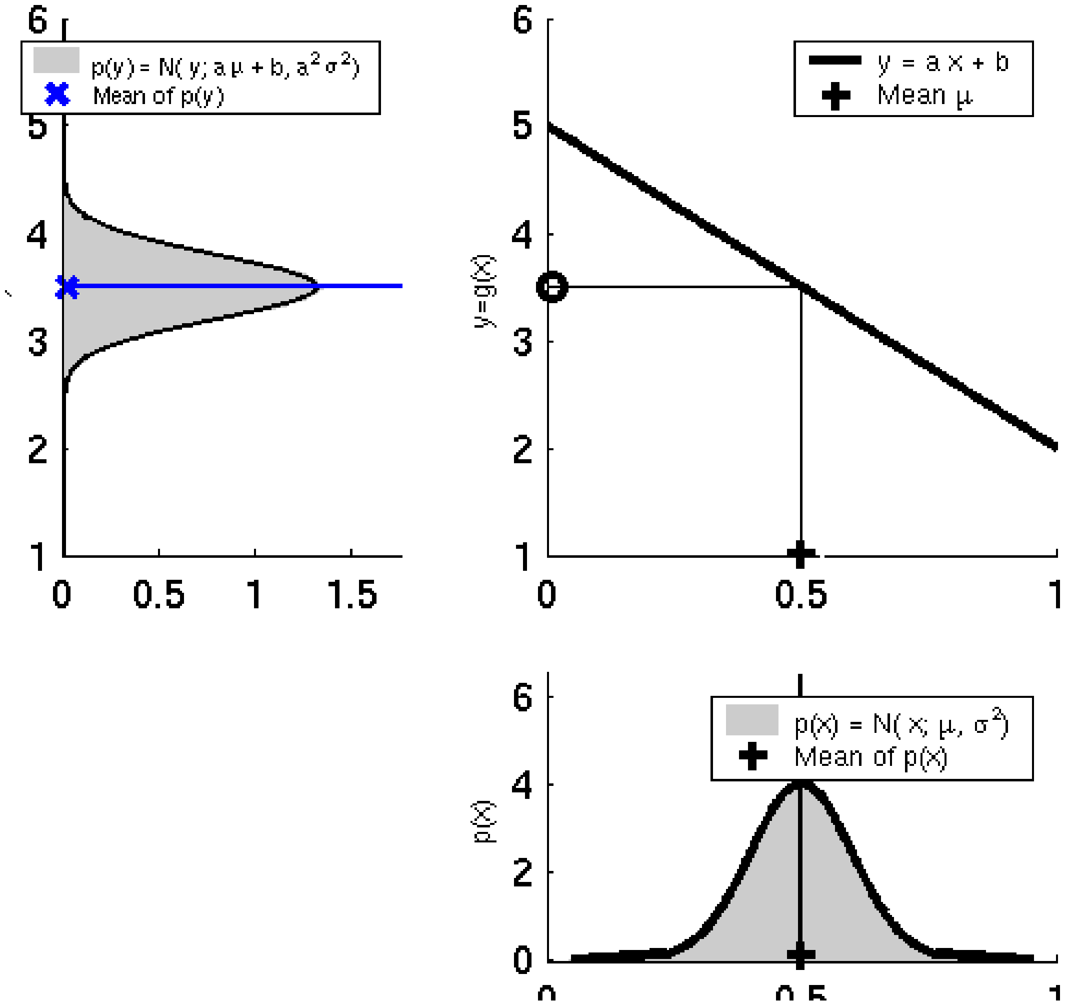
$$= \int p(x_t \mid x_{t-1}, u_t)\, bel(x_{t-1})\, dx_{t-1}$$

Discrete Bayes' filter

- Discrete Bayes' filter algorithm: applies to problems with *finite* state spaces
- Belief $bel(x_t)$ represented as pmf $\{p_{k,t}\}$

Data: $\{p_{k,t-1}\}, u_t, z_t$
Result: $\{p_{k,t}\}$
foreach k **do**
$\quad \bar{p}_{k,t} = \sum_i p(X_t = x_k \,|\, u_t, X_{t-1} = x_i)\, p_{i,t-1};$
$\quad p_{k,t} = \eta\, p(z_t \,|\, X_t = x_k)\, \bar{p}_{k,t};$
end
Return $\{p_{k,t}\}$

Next time



6
5
4
3
2
1
0
0.5
1
1.5
p(y) = N(y; a μ + b, a² σ²)
Mean of p(y)
y = a x + b
Mean μ
y=g(x)
p(x) = N(x; μ, σ²)
Mean of p(x)
p(x)