Principles of Robot Autonomy I

Markov localization and EKF-localization

Attendance Form

Agenda

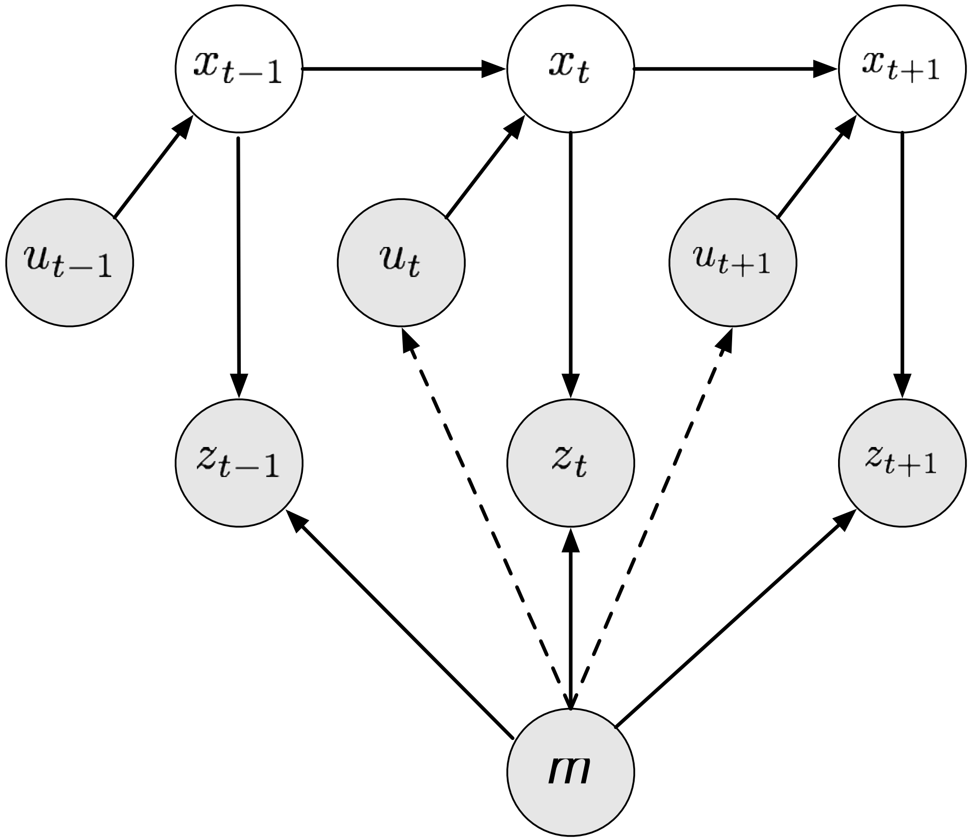
- Aim
  - Markov localization, with an emphasis on EKF localization

- Readings
  - Chapter 15, sections 15.1 – 15.4 in D. Gammelli, J. Lorenzetti, K. Luo, G. Zardini, M. Pavone. *Principles of Robot Autonomy*. 2026.

Mobile robot localization

- Problem: determine pose of a robot relative to a *given* map

- Localization can be interpreted as the problem of establishing correspondence between the map coordinate system and the robot's local coordinate frame
- This process requires integration of data over time



Local versus global localization

- Position tracking assumes that the initial pose is known -> *local* problem well-addressed via Gaussian filters
- In global localization, the initial pose is unknown -> *global* problem best addressed via non-parametric, multi-hypothesis filters
- In kidnapped robot localization, initial pose is unknown and during operation robot can be “kidnapped” and “teleported” to some other location -> *global* problem best addressed via non-parametric, multi-hypothesis filters

Static versus dynamic environments

- Static environments are environments where the only variable quantity is the pose of the robot
- Dynamic environments possess objects (e.g., people) other than the robot whose locations change over time -> addressed via either state augmentation or outlier rejection

Passive versus active localization

- In passive localization, localization module only *observes* the robot; i.e., robot's motion is not aimed at facilitating localization
- In active localization, robot's actions are aimed at minimizing the localization error
- Hybrid approaches are possible

Single-robot versus multi-robot

- In single-robot localization, a single, individual robot is involved in the localization process
- In multi-robot localization, a team of robots is engaged with localization, possibly cooperatively

In this class we will focus on local, static (or quasi-static), passive, single-robot localization problems

Casting the localization problem within a Bayesian filtering framework

- State x_t, control u_t and measurements z_t have the same meaning as in the general filtering context
- For a differential drive robot equipped with a laser range-finder (returning a set of range r_i and bearing ϕ_i measurements)

$$x_t = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \qquad u_t = \begin{pmatrix} v \\ \omega \end{pmatrix} \qquad z_t = \left\{ \begin{pmatrix} r_i \\ \phi_i \end{pmatrix} \right\}_i$$

Casting the localization problem within a Bayesian filtering framework

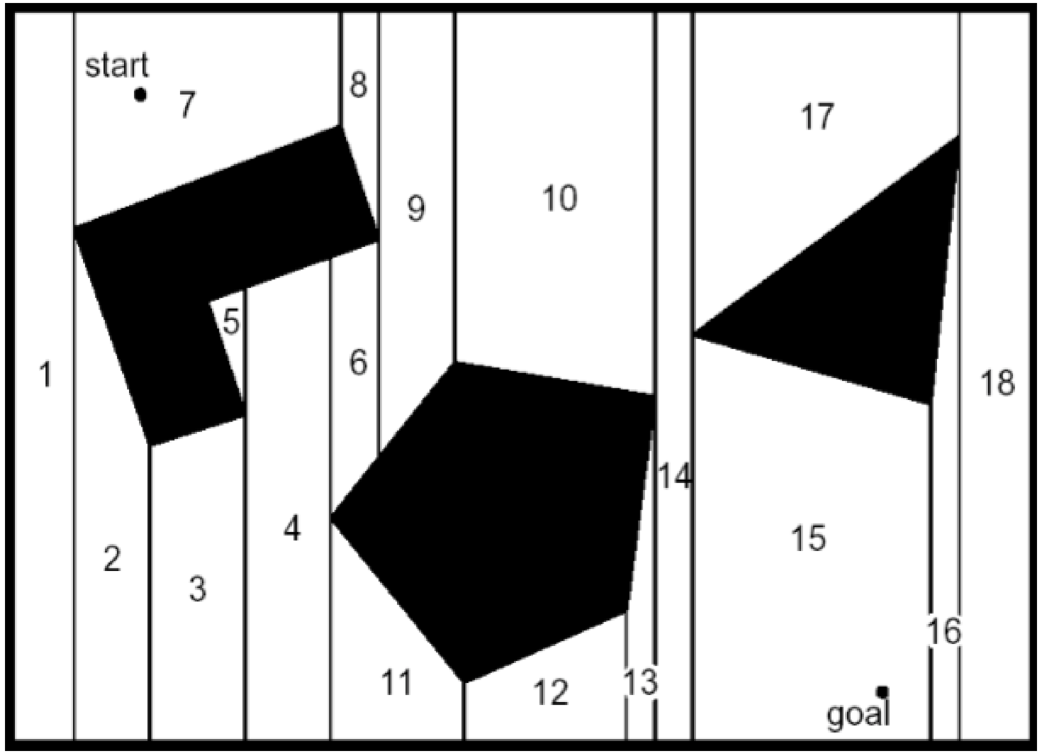
- A map m is a list of objects in the environment along with their properties

$$m = \{m_1, m_2, \ldots, m_N\}$$

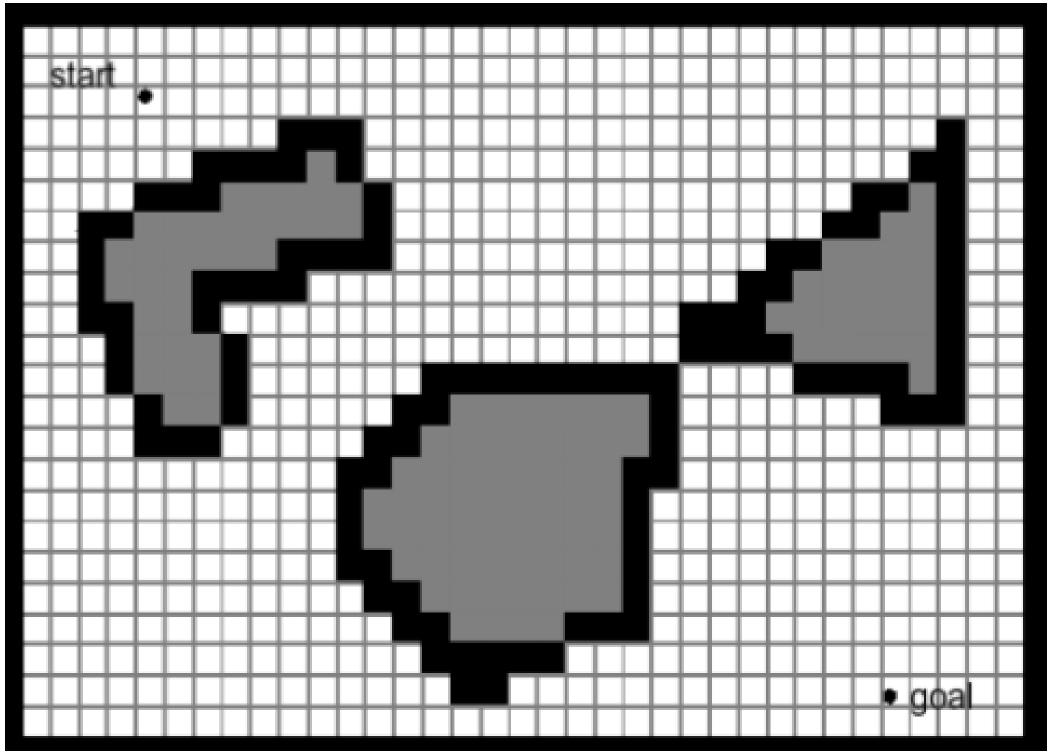
- Maps can be
  - *Location-based*: index i corresponds to a specific location (hence, they are volumetric)
  - *Feature-based*: index i is a feature index, and m_i contains, next to the properties of a feature, the Cartesian location of that feature

Location-based maps

Vertical cell decomposition

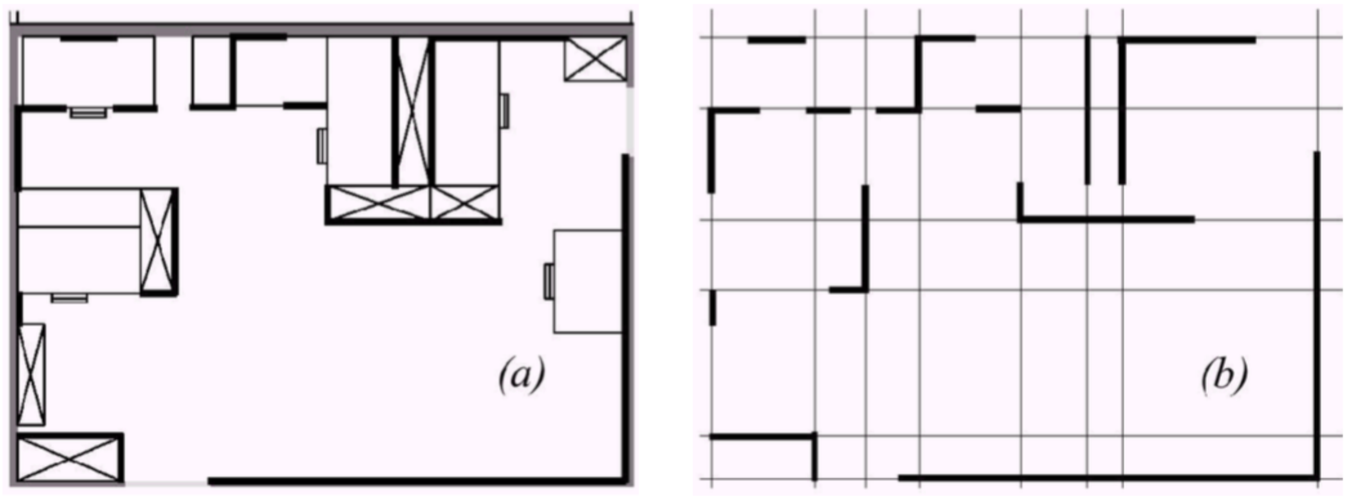


Fixed cell decomposition (occupancy grid)

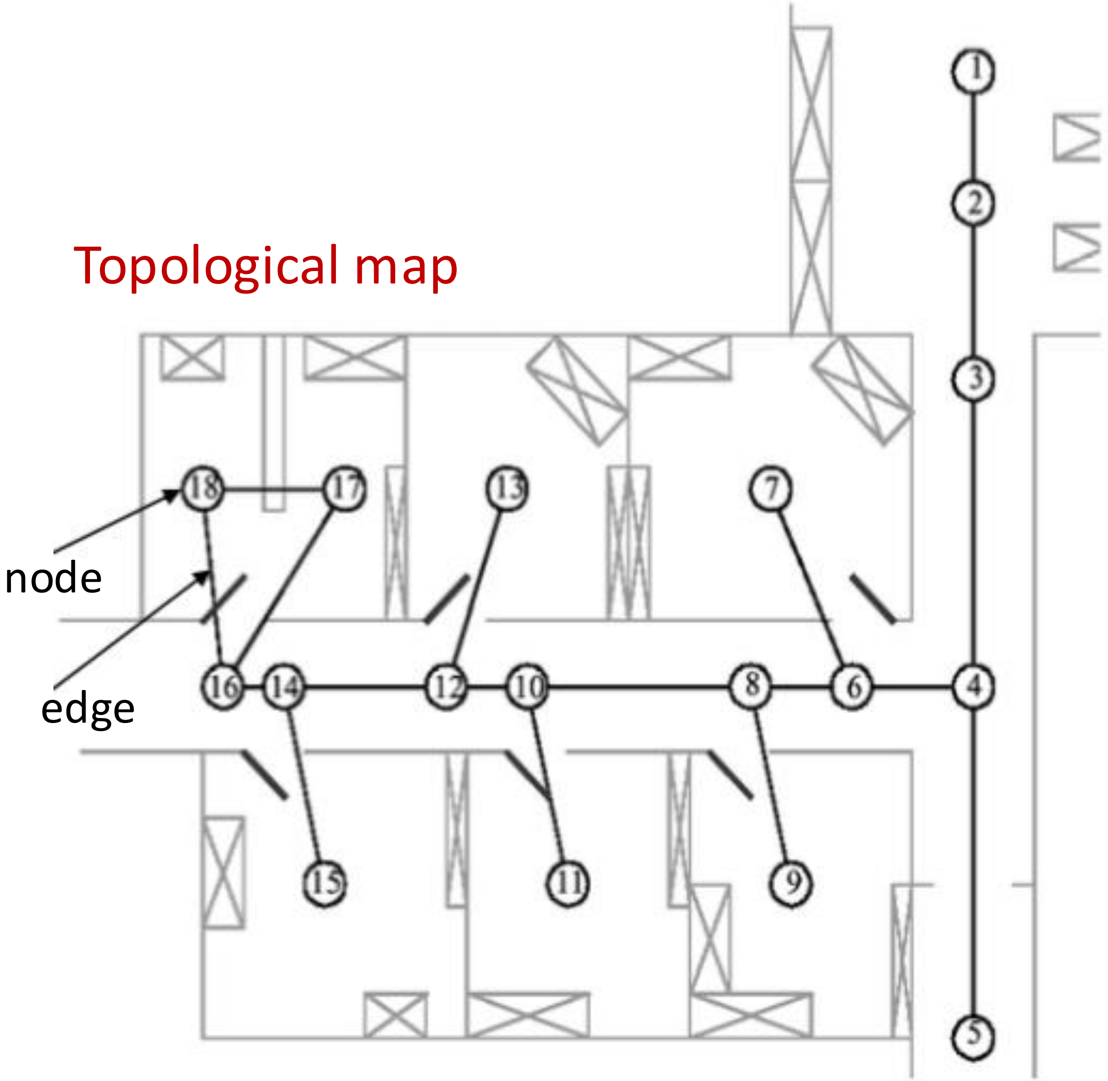


Feature-based maps

Line-based map

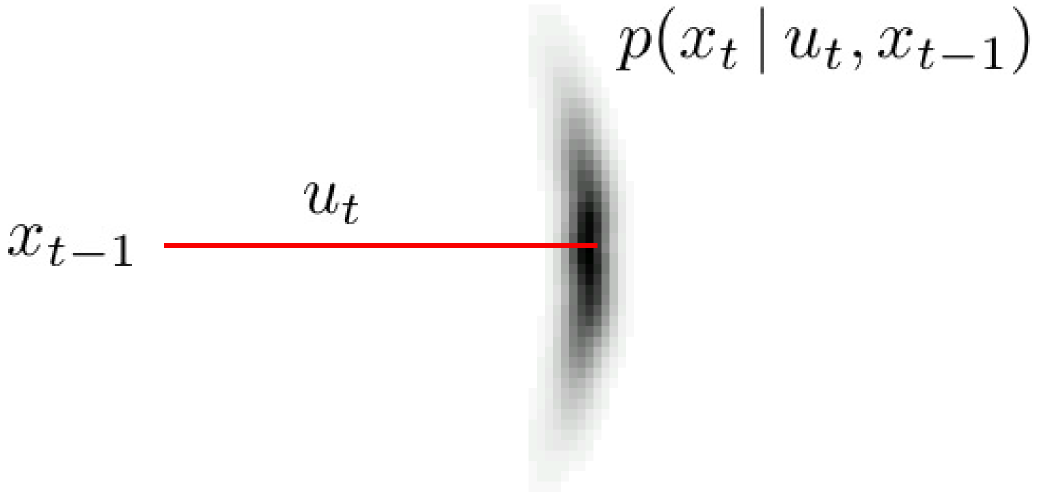


Topological map



Casting the localization problem within a Bayesian filtering framework

- Motion model is probabilistic



- Key fact: $p(x_t \,|\, u_t, x_{t-1}) \neq p(x_t \,|\, u_t, x_{t-1}, m)$
- Useful approximation (tight at high update rates)

$$p(x_t \,|\, u_t, x_{t-1}, m) \approx \eta \frac{p(x_t \,|\, u_t,\, x_{t-1})\, p(x_t \,|\, m)}{p(x_t)}$$

Consistency of state x_t with map m

Uses approximation
$p(m \,|\, x_t, u_t, x_{t-1}) \approx p(m \,|\, x_t)$

Casting the localization problem within a Bayesian filtering framework

- Measurement model is probabilistic

$$p(z_t \mid x_t, m)$$

- Sensors usually generate more than one measurement when queried

$$z_t = \{z_t^1, \ldots, z_t^K\}$$

- Typically, independence assumption is made

$$p(z_t \mid x_t, m) = \prod_{k=1}^{K} p(z_t^k \mid x_t, m)$$

Markov localization

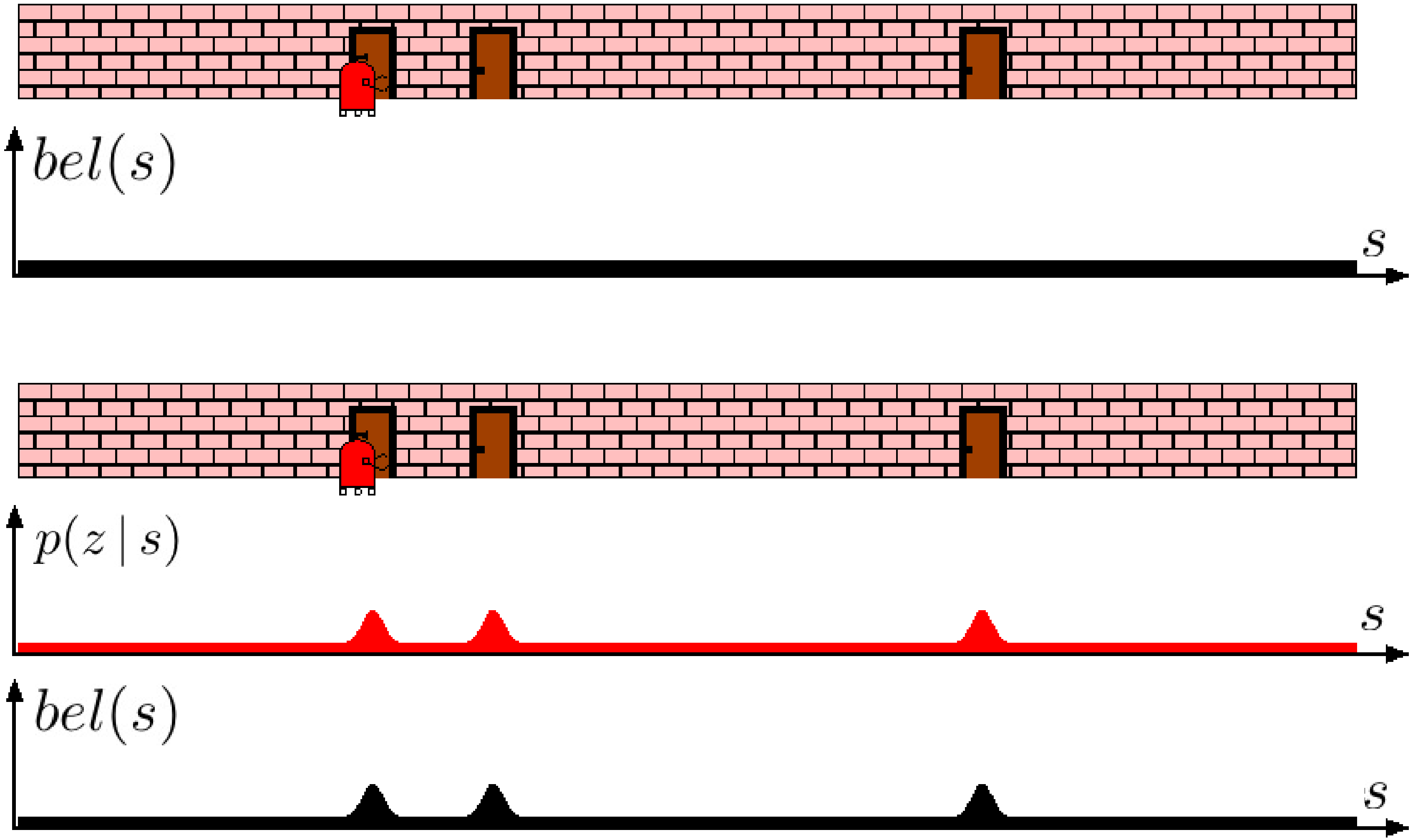
- Straightforward application of Bayes filter
- Requires a map *m* as input
- Addresses:
  - Position tracking
  - Global localization
  - Kidnapped robot problem

Data: $bel(x_{t-1}), u_t, z_t, m$
Result: $bel(x_t)$
foreach x_t **do**
$\quad \overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1}, m)\, bel(x_{t-1})\, dx_{t-1};$
$\quad bel(x_t) = \eta p(z_t \mid x_t, m)\, \overline{bel}(x_t);$
end
Return $bel(x_t)$

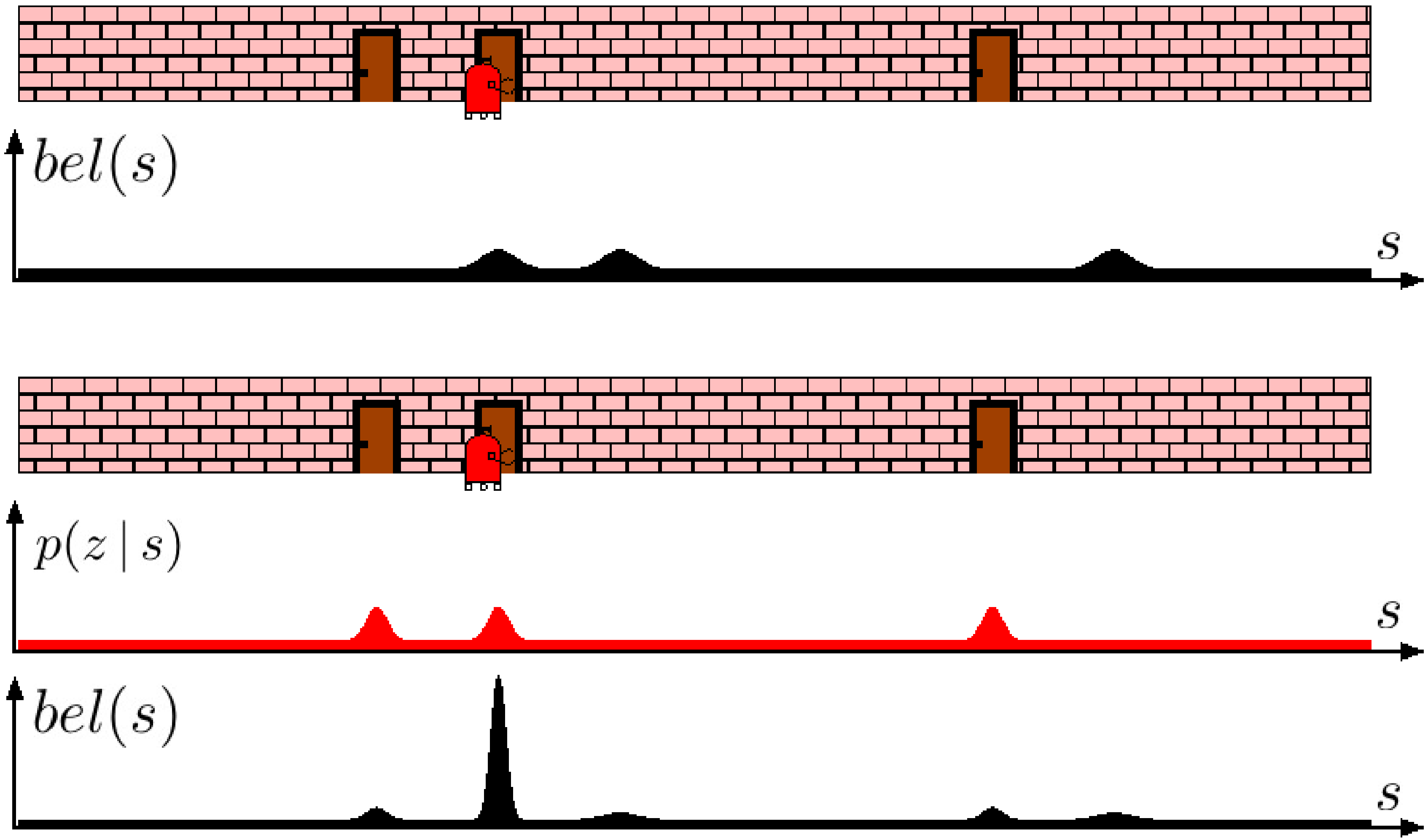
Markov localization: typical choices for initial belief

- Initial belief, $bel(x_0)$ reflects initial knowledge of robot pose
- For position tracking
  - If initial pose is known, $bel(x_0) = \begin{cases} 1 \text{ if } x_0 = \overline{x}_0 \\ 0 \text{ otherwise} \end{cases}$
  - If partially known, $bel(x_0) \sim \mathcal{N}(\overline{x}_0, \Sigma_0)$
- For global localization
  - If initial pose is unknown, $bel(x_0) = 1/|X|$

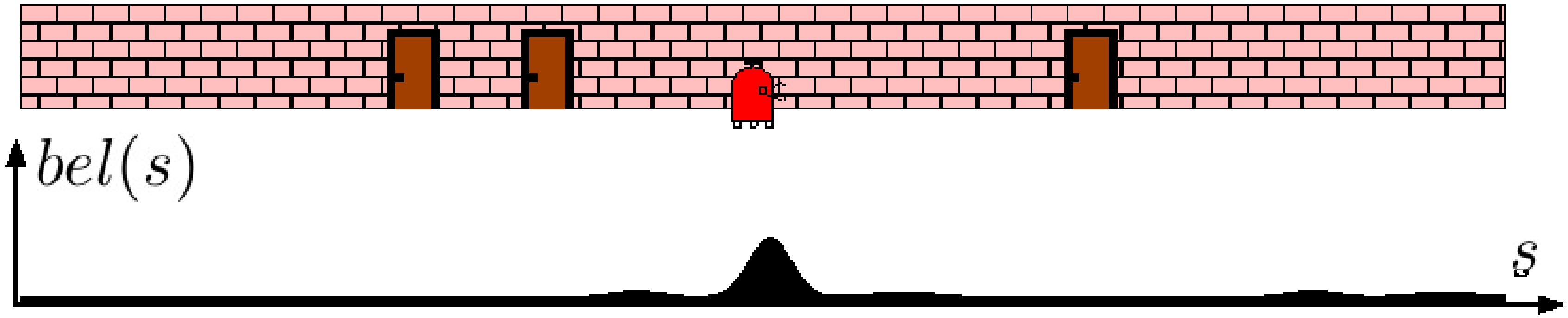
Markov localization: example



Markov localization: example



Markov localization: example



Instantiation of Markov localization

- To make algorithm tractable, we need to add some structure to the representation of $bel(x_t)$; examples:
  1. Gaussian representation <- focus of the rest of this lecture
  2. Particle filter representation

Extended Kalman filter (EKF) localization

- Key idea: represent belief $bel(x_t)$ by its first and second moment, i.e., μ_t and Σ_t
- We will develop the EKF localization algorithm under the assumptions that:
  1. A feature-based map is available, consisting of point landmarks

$$m = \{m_1, m_2, \ldots\}, \qquad m_i = (m_{i,x}, m_{i,y})$$

Location of the landmark in the global coordinate frame

  2. There is a sensor that can measure the range r and the bearing ϕ of the landmarks relative to the robot's local coordinate frame
- Key concepts carry forward to other map / sensing models

Range and bearing sensors

- Range & bearing sensors are common: features extracted from range scans and stereo vision come with range r and bearing ϕ information
- At time *t*, a set of features is measured (assumed independent)

$$z_t = \{z_t^1, z_t^2, \ldots\} = \{(r_t^1, \phi_t^1), (r_t^2, \phi_t^2), \ldots\}$$

- Assuming that the *i*-th measurement at time *t* corresponds to the *j*-th landmark in the map, the measurement model is

$$\begin{pmatrix} r_t^i \\ \phi_t^i \end{pmatrix} = \underbrace{\begin{pmatrix} \sqrt{(m_{j,x} - x)^2 + (m_{j,y} - y)^2} \\ \operatorname{atan2}(m_{j,y} - y, m_{j,x} - x) - \theta \end{pmatrix}}_{=h(x_t, j, m)} + \mathcal{N}(0, Q_t)$$

Gaussian noise

The issue of data association

- Data association problem: uncertainty may exists regarding the identity of a landmark
- Formally, we define a *correspondence variable* between measurement z_t^i and landmark m_j in the map as (assume N landmarks)

$$c_t^i \in \{1, \ldots, N+1\}$$

  - $c_t^i = j \leq N$ if *i*-th measurement at time *t* corresponds to *j*-th landmark
  - $c_t^i = N+1$ if a measurement does not correspond to any landmark
- Two versions of the localization problem
  1. Correspondence variables are known
  2. Correspondence variables are not known (usual case)

EKF localization with known correspondences

- Algorithm is derived from EKF filter
- Assume motion model (in our case, differential drive robot)

$$x_t = g(u_t, x_{t-1}) + \epsilon_t, \qquad \epsilon_t \sim \mathcal{N}(0, R_t), \qquad G_t := J_g(u_t, \mu_{t-1})$$

- Assume range and bearing measurement model

$$z_t^i = h(x_t, j, m) + \delta_t, \qquad \delta_t \sim \mathcal{N}(0, Q_t), \qquad H_t^i := \frac{\partial h(\overline{\mu}_t, j, m)}{\partial x_t}$$

$$\frac{\partial h(\overline{\mu}_t, j, m)}{\partial x_t} = \begin{pmatrix} \frac{\partial r_t^i}{\partial \overline{\mu}_{t,x}} & \frac{\partial r_t^i}{\partial \overline{\mu}_{t,y}} & \frac{\partial r_t^i}{\partial \overline{\mu}_{t,\theta}} \\ \frac{\partial \phi_t^i}{\partial \overline{\mu}_{t,x}} & \frac{\partial \phi_t^i}{\partial \overline{\mu}_{t,y}} & \frac{\partial \phi_t^i}{\partial \overline{\mu}_{t,\theta}} \end{pmatrix} = \begin{pmatrix} -\frac{m_{j,x} - \overline{\mu}_{t,x}}{\sqrt{(m_{j,x} - \overline{\mu}_{t,x})^2 + (m_{j,y} - \overline{\mu}_{t,y})^2}} & -\frac{m_{j,y} - \overline{\mu}_{t,y}}{\sqrt{(m_{j,x} - \overline{\mu}_{t,x})^2 + (m_{j,y} - \overline{\mu}_{t,y})^2}} & 0 \\ \frac{m_{j,y} - \overline{\mu}_{t,y}}{(m_{j,x} - \overline{\mu}_{t,x})^2 + (m_{j,y} - \overline{\mu}_{t,y})^2} & -\frac{m_{j,x} - \overline{\mu}_{t,x}}{(m_{j,x} - \overline{\mu}_{t,x})^2 + (m_{j,y} - \overline{\mu}_{t,y})^2} & -1 \end{pmatrix}$$

$$Q_t = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\phi^2 \end{pmatrix}$$

EKF localization with known correspondences

- Main difference with EKF filter: multiple measurements are processed at the same time
- We exploit conditional independence assumption

$$p(z_t \mid x_t, c_t, m) = \prod_i p(z_t^i \mid x_t, c_t^i, m)$$

- Such assumption allows us to incrementally add the information, as if there was zero motion in between measurements

Data: $(\mu_{t-1}, \Sigma_{t-1}), u_t, z_t, c_t, m$
Result: (μ_t, Σ_t)
$\bar{\mu}_t = g(u_t, \mu_{t-1})$;
$\bar{\Sigma}_t = G_t \Sigma_{t-1} {G_t}^T + R_t$;

foreach $z_t^i = (r_t^i, \phi_t^i)^T$ **do**

$j = c_t^i$;

$$\hat{z}_t^i = \begin{pmatrix} \sqrt{(m_{j,x} - \bar{\mu}_{t,x})^2 + (m_{j,y} - \bar{\mu}_{t,y})^2} \\ \text{atan2}(m_{j,y} - \bar{\mu}_{t,y}, m_{j,x} - \bar{\mu}_{t,x}) - \bar{\mu}_{t,\theta} \end{pmatrix};$$

$S_t^i = H_t^i \, \bar{\Sigma}_t \, [H_t^i]^T + Q_t$;

Innovation covariance

$K_t^i = \bar{\Sigma}_t \, [H_t^i]^T \, [S_t^i]^{-1}$;
$\bar{\mu}_t = \bar{\mu}_t + K_t^i (z_t^i - \hat{z}_t^i)$;
$\bar{\Sigma}_t = (I - K_t^i H_t^i) \, \bar{\Sigma}_t$;

end

$\mu_t = \bar{\mu}_t$ and $\Sigma_t = \bar{\Sigma}_t$;
Return (μ_t, Σ_t)

EKF localization with unknown correspondences

- Key idea: determine the identity of a landmark during localization via maximum likelihood estimation, whereby one first determines the most likely value of c_t, and then takes this value for granted
- Formally, the maximum likelihood estimator determines the correspondence that maximizes the data likelihood

$$\hat{c}_t = \arg\max_{c_t} p(z_t \mid c_{1:t}, m, z_{1:t-1}, u_{1:t})$$

- Challenge: there are exponentially many terms in the maximization above!
- Solution: perform maximization *separately* for each z_t^i

Estimating the correspondence variables

- Step #1: find

$$p(z_t^i \mid c_{1:t}, m, z_{1:t-1}, u_{1:t})$$

- Derivation (sketch)

$$p(z_t^i \mid c_{1:t}, m, z_{1:t-1}, u_{1:t}) = \int p(z_t^i \mid x_t, c_{1:t}, m, z_{1:t-1}, u_{1:t})\, p(x_t \mid c_{1:t}, m, z_{1:t-1}, u_{1:t})\, dx_t$$

$$= \int p(z_t^i \mid x_t, c_t^i, m) \quad \cdot \quad \overline{bel}(x_t)\, dx_t$$

$$\sim \mathcal{N}(h(x_t, c_t^i, m), Q_t) \qquad\qquad \sim \mathcal{N}(\overline{\mu}_t, \overline{\Sigma}_t)$$

$$\approx \mathcal{N}(h(\overline{\mu}_t, c_t^i, m) + H_t^i(x_t - \overline{\mu}_t), Q_t)$$

Estimating the correspondence variables

- Performing the algebraic calculations

$$p(z_t^i \,|\, c_{1:t}, m, z_{1:t-1}, u_{1:t}) \approx \mathcal{N}(h(\overline{\mu}_t, c_t^i, m), H_t^i\, \overline{\Sigma}_t\, [H_t^i]^T + Q_t)$$

- Step #2: estimate correspondence as

$$\begin{aligned}\hat{c}_t^i &= \arg\max_{c_t^i}\, p(z_t^i | c_{1:t}, m, z_{1:t-1}, u_{1:t}) \\ &\approx \arg\max_{c_t^i}\, \mathcal{N}(z_t^i;\, h(\bar{\mu}_t, c_t^i, m), H_t \bar{\Sigma}_t H_t^T + Q_t)\end{aligned}$$

EKF localization with unknown correspondences

- Same as before, plus the inclusion of a maximum likelihood estimator for the correspondence variables

Data: $(\mu_{t-1}, \Sigma_{t-1}), u_t, z_t, m$
Result: (μ_t, Σ_t)
$\overline{\mu}_t = g(u_t, \mu_{t-1})$;
$\overline{\Sigma}_t = G_t \Sigma_{t-1} {G_t}^T + R_t$;

foreach $z_t^i = (r_t^i, \phi_t^i)^T$ **do**
  foreach *landmark k in the map* **do**

$$\hat{z}_t^k = \begin{pmatrix} \sqrt{(m_{k,x} - \overline{\mu}_{t,x})^2 + (m_{k,y} - \overline{\mu}_{t,y})^2} \\ \text{atan2}(m_{k,y} - \overline{\mu}_{t,y}, m_{k,x} - \overline{\mu}_{t,x}) - \overline{\mu}_{t,\theta} \end{pmatrix};$$

$$S_t^k = H_t^k \, \overline{\Sigma}_t \, [H_t^k]^T + Q_t;$$

  end

$$j(i) = \arg\max_k \mathcal{N}(z_t^i; \hat{z}_t^k, S_t^k)$$

$$K_t^i = \overline{\Sigma}_t \, [H_t^{j(i)}]^T \, [S_t^{j(i)}]^{-1};$$

$$\overline{\mu}_t = \overline{\mu}_t + K_t^i (z_t^i - \hat{z}_t^{j(i)});$$

$$\overline{\Sigma}_t = (I - K_t^i H_t^{j(i)}) \, \overline{\Sigma}_t;$$

end
$\mu_t = \overline{\mu}_t$ and $\Sigma_t = \overline{\Sigma}_t$;
Return (μ_t, Σ_t)

Correspondence estimation

Next time